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Optimal Sampling of Periodic Analytic Functions

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Let $S=\{z\in\mathbb{C}: |\mathrm{Im}(z)|<\beta\}$ be a strip in the complex plane. $\tilde{H}^2$ denotes the space of functions f, which are analytic and 2π-periodic in S and satisfy

$$\|f\|_{\tilde{H}^2} := \sup_{0\le\eta<\beta}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t+i\eta)|^2+|f(t-i\eta)|^2\,dt\right)^{1/2}<\infty.$$

The Kolmogorov n-widths d_n, Gel'fand n-widths d^n, and linear n-widths δ_n of $\tilde{H}^2$ in $\tilde{L}_2$, the periodic Lebesgue space on the real axis are determined by

$$d_{2n-1}(\tilde{H}^2,\tilde{L}_2)=d_{2n}(\tilde{H}^2,\tilde{L}_2)=\left(\frac{1}{2\cosh(2n\beta)}\right)^{1/2}.$$

The same equations hold for $d^n(\tilde{H}^2,\tilde{L}_2)$ and $\delta_n(\tilde{H}^2,\tilde{L}_2)$. Fourier expansion of order $2n-1$ is an optimal linear approximation operator for $\delta_{2n-1}=\delta_{2n}$. In addition, we construct an optimal linear $2n$-dimensional approximation method, which is based on sampling a function $f\in\tilde{H}^2$ in $2n$ equidistant points in $[0,2\pi]$.

1. INTRODUCTION

Let $S=\{z\in\mathbb{C}: |\mathrm{Im}(z)|<\beta\}$ be a strip in the complex plane. In the present paper we study the n-widths of the space $\tilde{H}^2$, consisting of all functions f, which are analytic and 2π-periodic in S and satisfy

$$\|f\|_{\tilde{H}^2} := \sup_{0\le\eta<\beta}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t+i\eta)|^2+|f(t-i\eta)|^2\,dt\right)^{1/2}<\infty.$$

A function f in $\tilde{H}^2$ has a non-tangential limit almost everywhere on ∂S. The boundary function belongs to $\tilde{L}_2(\partial S)$ and the scalar product

$$(f,g)_{\tilde{H}^2}=\frac{1}{2\pi}\int_0^{2\pi}f(t+i\beta)\,\overline{g(t+i\beta)}\,dt+\frac{1}{2\pi}\int_0^{2\pi}f(t-i\beta)\,\overline{g(t-i\beta)}\,dt$$

induces a Hilbert space structure on $\tilde{H}^2$.

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We find the exact value of the n-widths of the unit ball of $\tilde{H}^2$ in $\tilde{L}_2$, the periodic complex-valued Lebesgue space on the real axis. Furthermore, we show that sampling in $2n$ equidistant points in $[0, 2\pi]$ yields an optimal $2n$-dimensional linear approximation operator. Finally an explicit representation for the optimal approximation operator is given in terms of elliptic functions.

Analogous results are already known for the n-widths of the space $\tilde{K}^2$, consisting of all functions f, which are analytic in S, real and 2π-periodic on the real axis, and satisfy

$$\sup_{0 \leq \eta < \beta} \frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} f(t + i\eta)|^2 \, dt < \infty.$$

The n-widths of the unit ball of $\tilde{K}^2$ in $\tilde{L}_2^{\mathbb{R}}$, the periodic real-valued Lebesgue space on the real axis, are given in Pinkus [7, Chap. IV. 6]. Pinkus also established the optimality of sampling for $\tilde{K}^2$.

The fundamental difference between $\tilde{H}^2$ and $\tilde{K}^2$ lies in the fact that functions in $\tilde{K}^2$ may be characterized as convolutions with a cyclic variation diminishing kernel, while such a representation is not available for functions in $\tilde{H}^2$. Therefore other techniques must be applied in order to deal with $\tilde{H}^2$. Our approach will consist in transfering the analysis from the strip S to the annulus $\Omega = \{w \in \mathbb{C} : q < |w| < 1/q\}$, where $q = e^{-\beta}$. Then we will study the equivalent problems for holomorphic functions defined on Ω. For this purpose we extend some techniques, which were developed by Fisher and Micchelli [3] for investigating the H^2-space in the unit disk, to the doubly connected annulus.

In Section 2 we formulate our main results, while Section 3 contains the corresponding proofs.

2. Main Results

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let us assume that X is continuously imbedded into Y by an imbedding operator $T: X \to Y$.

The Kolmogorov n-widths d_n of X in Y are defined by

$$d_n(X, Y) = \inf_{Y_n} \sup_{\|x\|_X \leq 1} \inf_{y \in Y_n} \|x - y\|_Y,$$

where Y_n runs over all subspaces of Y of dimension n or less.

The Gel'fand n-widths of X in Y are defined by

$$d^n(X, Y) = \inf_{X_n} \sup_{\substack{\|x\|_X \leq 1 \\ x \in X_n}} \|x\|_Y,$$

where the infimum is taken over all subspaces X_n of X of codimension at most n.

The linear n-widths δ_n of X in Y are given by

$$\delta_n(X, Y) = \inf_{P_n} \sup_{\|x\|_X \leq 1} \|x - P_n x\|_Y,$$

where P_n is any continuous linear operator of X into Y of rank at most n.

The aim of the present paper is the investigation of the n-widths of the imbedding

$$\tilde{T}: \tilde{H}^2 \to \tilde{L}_2.$$

Our approach to this problem will consist in transfering the analysis from the strip S to the annulus $\Omega = \{w \in \mathbb{C} : q < |w| < 1/q\}$, where $q = e^{-\beta}$. The transformation $w = e^{iz}$ maps S onto Ω and the operator

$$I: f(z) \to g(w) = f\left(\frac{1}{i} \log(w)\right)$$

yields an isometry between $\tilde{H}^2$ and H^2, the space of all functions g, which are analytic in Ω and possess square integrable boundary values:

$$\|g\|_{H^2} := \left(\frac{1}{2\pi} \int_0^{2\pi} |g(qe^{i\theta})|^2 \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \left| g\left(\frac{1}{q} e^{i\theta}\right)\right|^2 d\theta\right)^{1/2} < \infty.$$

The trigonometric polynomials $\{e^{ikz}\}_{k \in \mathbb{Z}}$ on S correspond to the monomials $\{w^k\}_{k \in \mathbb{Z}}$ on Ω and the Fourier coefficients of $f \in \tilde{H}^2$ are equal to the Laurent coefficients of $If \in H^2$. Furthermore, I maps $\tilde{L}_2$ isometrically onto the space $L_2(E)$, where $E = \{w \in \mathbb{C} : |w| = 1\}$ represents the unit circle. Denoting by T the imbedding operator from H^2 into $L_2(E)$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}^2 & \xrightarrow{\tilde{T}} & \tilde{L}_2 \\ {\scriptstyle I}\updownarrow & & \updownarrow{\scriptstyle I} \\ H^2 & \xrightarrow[T]{} & L_2(E) \end{array}$$

From the diagram we see that the n-widths of $\tilde{H}^2$ in $\tilde{L}_2$ are equal to the n-widths of H^2 in $L_2(E)$. Every optimal subspace for $\tilde{T}$ yields an optimal subspace for T, and vice versa. The same is true for optimal linear approximation operators. In the following we will formulate all our results for the imbedding $T: H^2 \to L_2(E)$ for technical convenience; the back-transformation from T to $\tilde{T}$ is straightforward. With this convention our first result reads as follows.

THEOREM 1.

$$d_{2n-1}(H^2, L_2(E)) = d_{2n}(H^2, L_2(E)) = \left(\frac{1}{q^{2n}+q^{-2n}}\right)^{1/2}.$$

The same equations hold for $d^n(H^2, L_2(E))$ and $\delta_n(H^2, L_2(E))$.
Furthermore, let us denote by

$$a_k(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, e^{-ik\theta}\, d\theta, \qquad k \in \mathbb{Z},$$

the Laurent coefficients of $f \in H^2$. Then we have

(i) $P_{2n-1}: H^2 \to L_2(E)\; f \mapsto \sum_{k=-(n-1)}^{n-1} a_k(f)\, e^{ik\theta}$ is an optimal linear operator for $\delta_{2n-1}(H^2, L_2(E)) = \delta_{2n}(H^2, L_2(E))$.

(ii) $X_{2n-1} = \{f \in H^2 : a_{-(n-1)}(f) = \cdots = a_{n-1}(f) = 0\}$ is an optimal subspace for $d^{2n-1}(H^2, L_2(E)) = d^{2n}(H^2, L_2(E))$.

(iii) $Y_{2n-1} = \operatorname{span}\{e^{-i(n-1)\theta}, ..., e^{i(n-1)\theta}\}$ is an optimal subspace for $d_{2n-1}(H^2, L_2(E)) = d_{2n}(H^2, L_2(E))$.

Theorem 1 will be proven by applying classical spectral methods. Our second main result states that in the even-dimensional case in addition to the classical methods there exist nonclassical optimal approximation methods, which are based on sampling. To fix our notation, let

$$\zeta_j = \exp\left((j-1)\, i\, \frac{2\pi}{2n}\right), \qquad j = 1, ..., 2n,$$

be the roots of unity of order $2n$. Let us define an information operator L by

$$L: H^2 \to \mathbb{C}^{2n} \qquad f \mapsto (f(\zeta_1), ..., f(\zeta_{2n})).$$

For $f \in H^2$ we define $\sigma_{2n}(f)$ to be the solution of the minimum norm problem

$$\min_{Lg = Lf} \|g\|_{H^2}.$$

Since H^2 is a Hilbert space, the last extremal problem attains a unique minimum, which depends linearly on f, i.e.,

$$\sigma_{2n}(f) = \sum_{j=1}^{2n} f(\zeta_j)\, h_j,$$

where the functions $h_j \in H^2$, $j = 1, \ldots, 2n$, are independent of f. Because of the minimum norm property $\sigma_{2n}(f)$ is called the spline interpolating $(f(\zeta_1), \ldots, f(\zeta_{2n}))$. We are now prepared to formulate

THEOREM 2.

(i) *Spline interpolation in* $2n$ *equidistant nodes is optimal:*

$$\sigma_{2n}\colon H^2 \to L_2(E) \qquad f \mapsto \sum_{j=1}^{2n} f(\zeta_j)\, h_j(e^{i\theta})$$

is an optimal linear operator for $\delta_{2n}(H^2, L_2(E))$.

(ii)

$$V^{2n} = \{ f \in H^2 : f(\zeta_j) = 0, j = 1, \ldots, 2n \}$$

is an optimal subspace for $d^{2n}(H^2, L_2(E))$.

(iii)

$$V_{2n} = \operatorname{span}\{ h_1(e^{i\theta}), \ldots, h_{2n}(e^{i\theta}) \}$$

is an optimal subspace for $d_{2n}(H^2, L_2(E))$.

The proof of Theorem 2 relies decisively on the fact that H^2 possesses a reproducing kernel, i.e., for each $\zeta \in \Omega$ there exists a function $K(\cdot, \zeta) \in H^2$ such that $f(\zeta) = (f(\cdot), K(\cdot, \zeta))_{H^2}$ for all $f \in H^2$.

With the help of elliptic functions it is possible to give an explicit representation for the spline interpolant. We will use the standard notation for the Jacobi elliptic functions $sn(z, k)$, $cn(z, k)$ and $dn(z, k)$ with modulus k (see for example Bateman [1]). The complementary modulus is given by $k' = \sqrt{1 - k^2}$ and the complete elliptic integrals of the first kind with moduli k and k' are denoted by K and K', respectively.

THEOREM 3. *The spline interpolant* $\sigma_{2n}(f)$ *for* $f \in H^2$ *is given explicitly by the formula*

$$\sigma_{2n}(f)(\zeta) = \frac{K}{2n\Lambda}\, sn\left(\frac{2n\Lambda}{\pi i} \log(\zeta), \lambda\right) \sum_{j=1}^{2n} (-1)^{j+1} \times \frac{cn((K/\pi i)\log(\zeta) - (j-1)(K/n), k)}{sn((K/\pi i)\log(\zeta) - (j-1)(K/n), k)} f(\zeta_j).$$

Here K *and* Λ *are complete elliptic integrals of the first kind with corresponding moduli* k *and* λ *determined by the equations*

$$\frac{\pi K'}{2K} = \beta \qquad \frac{\Lambda'}{\Lambda} = 2n \frac{K'}{K}.$$

COROLLARY. *For functions $f \in \tilde{H}^2$, analytic and 2π-periodic in the strip S, the optimal spline interpolant has the form*

$$\sigma_{2n}(f)(z) = \frac{K}{2n\Lambda} sn\left(\frac{2n\Lambda}{\pi} z, \lambda\right) \sum_{j=1}^{2n} (-1)^{j+1} \times \frac{cn((K/\pi)z - (j-1)(K/n), k)}{sn((K/\pi)z - (j-1)(K/n), k)} f\left(\frac{(j-1)\pi}{n}\right).$$

Theorem 3 was brought to the author's attention by the referee. The author is grateful to the referee for his valuable advice.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Let us start with the fundamental observation that the monomials $\{z^n\}_{n \in \mathbb{Z}}$ form a complete orthogonal system in H^2 (cf. Sarason [8]). Since

$$\|z^n\|_{H^2}^2 = q^{2n} + q^{-2n},$$

the unit ball U in H^2 can be characterized in the following way:

$$U = \{f \in H^2 : \|f\|_{H^2} \leq 1\} = \left\{f = \sum_{k=-\infty}^{\infty} a_k z^k : \left(\sum_{k=-\infty}^{\infty} |a_k|^2 (q^{2k} + q^{-2k})\right)^{1/2} \leq 1\right\}$$

Denoting the imbedding operator $H^2 \to L_2(E)$ by T we obtain

$$T(U) = \left\{f = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} : \left(\sum_{k=-\infty}^{\infty} |a_k|^2 (q^{2k} + q^{-2k})\right)^{1/2} \leq 1\right\}.$$

Since $\{e^{ik\theta}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $L_2(E)$, we see that $T(U)$ is an "ellipsoid" in $L_2(E)$. Thus Kolmogorov's theory of n-widths of ellipsoids (see Lorentz [5], Pinkus [7, Chap. IV. 2]) yields that

$$d_{2n-1}(H^2, L_2(E)) = d_{2n}(H^2, L_2(E)) = \left(\frac{1}{q^{2n} + q^{-2n}}\right)^{1/2}$$

and that $\text{span}\{e^{-i(n-1)\theta}, ..., e^{i(n-1)\theta}\}$ is an optimal $(2n-1)$-dimensional subspace.

The remaining assertions of Theorem 1 follow from the fact that both H^2 and $L_2(E)$ are Hilbert spaces and therefore

$$\delta_n(H^2, L_2(E)) = d^n(H^2, L_2(E)) = d_n(H^2, L_2(E)).$$

For further details we refer the reader to Pinkus [7, Chap. IV. 2]. In conclusion, we stress once more the decisive point of the Proof of Theorem 1, namely that the monomials are at the same time a complete orthogonal system in H^2 as well as in $L_2(E)$.

Proof of Theorem 2. Our approach to Theorem 2 uses ideas similar to those of Fisher and Micchelli [3], who investigated the H^2-space on the unit disk (see also Fisher and Micchelli [4] and Fisher [2]). It turns out that some of their techniques can be extended from the unit disk to the doubly connected annulus.

In order to prove Theorem 2 we have to show that

$$\sup_{f \in U} \|f - \sigma_{2n}(f)\|_{L_2(E)} = \left(\frac{1}{q^{2n} + q^{-2n}}\right)^{1/2}.$$

Here $\sigma_{2n}(f) = \sum_{j=1}^{2n} f(\zeta_j)\, h_j \in H^2$ is the spline, which interpolates the data $(f(\zeta_1), ..., f(\zeta_{2n}))$ with minimal H^2-norm. Since H^2 is a Hilbert space, $\sigma_{2n}(f)$ is well defined, it depends linearly on the data, and $Q_{2n}: f \mapsto f - \sigma_{2n}(f)$ is the orthogonal projection of H^2 onto the subspace $V^{2n} = \{f \in H^2 : f(\zeta_j) = 0, j = 1, ..., 2n\}$. Thus

$$\begin{aligned} &\sup_{f \in U} \|f - \sigma_{2n}(f)\|_{L_2(E)} \\ &\quad = \sup_{f \in U} \|Q_{2n} f\|_{L_2(E)} = \sup\{\|f\|_{L_2(E)} : f \in U, f(\zeta_j) = 0, j = 1, ..., 2n\} =: \delta. \end{aligned}$$

As indicated above, we will denote the last supremum by δ in the sequel. δ may be characterized with the help of the Blaschke product B on the annulus Ω with zeros in $\zeta_1, ..., \zeta_{2n}$, which is defined as follows:

$$B(z) = \varepsilon \exp\left(-\sum_{j=1}^{2n} P(z, \zeta_j)\right).$$

Here ε is a constant factor of modulus one and

$$P(z, \zeta_j) = g(z, \zeta_j) + ih(z, \zeta_j),$$

where $g(z, \zeta_j)$ is the Green's function for Ω with singularity in ζ_j and $h(z, \zeta_j)$ is the harmonic conjugate of $g(z, \zeta_j)$. Although each $h(z, \zeta_j)$ is not single-valued, $B(z)$ is single-valued, because the period of the sum $\sum_{j=1}^{2n} h(z, \zeta_j)$ is an integer multiple of 2π. Furthermore, we may assume after scaling that B is real-valued on E. For more details on Blaschke products we refer the reader to Fisher [2].

Of decisive importance is the fact that $|B(z)| = 1$ for $z \in \partial\Omega$, since $g(z, \zeta_j)$ vanishes identically on $\partial\Omega$. This property implies that

$$\delta = \sup\left\{\left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, |B(e^{i\theta})|^2\, d\theta\right)^{1/2} : f \in U\right\}.$$

This motivates us to introduce on E the measure

$$d\mu(\theta) = |B(e^{i\theta})|^2\, d\theta.$$

In the following we must carefully distinguish between the Lebesgue space $L_2(E)$ with respect to the standard measure $d\theta$ and the space $L_2(E, \mu)$ with respect to the measure μ. With this notation we obtain that

$$\delta = \sup\{\, \|f\|_{L_2(E,\mu)} : f \in U\} = \|\tau\colon H^2 \to L_2(E, \mu)\|,$$

where τ denotes the imbedding operator from H^2 into $L_2(E, \mu)$.

In view of the fact that $\|\tau\|^2 = \|\tau\tau'\|$, where $\tau'\colon L_2(E, \mu) \to H^2$ denotes the adjoint operator of τ, our further strategy consists in determining the largest eigenvalue of the selfadjoint, nonnegative, compact operator $\tau\tau'\colon L_2(E, \mu) \to L_2(E, \mu)$. For this purpose we use heavily the reproducing kernel structure of H^2.

Since the normalized monomials $\varphi_k = z^k/(q^{2k} + q^{-2k})^{1/2}$, $k \in \mathbb{Z}$, are an orthonormal basis of H^2, the reproducing kernel is given by

$$K(z, \zeta) = \sum_{k=-\infty}^{\infty} \varphi_k(z)\, \overline{\varphi_k(\zeta)} = \sum_{k=-\infty}^{\infty} \frac{z^k \bar{\zeta}^k}{q^{2k} + q^{-2k}}. \tag{2.1}$$

Now let λ be an eigenvalue of $\tau\tau'\colon L_2(E, \mu) \to L_2(E, \mu)$ with eigenfunction ϕ:

$$\tau\tau'\phi = \lambda\phi. \tag{2.2}$$

Since by definition $\tau f = f|_E$, (2.2) can be written in the form

$$(\tau'\phi)(e^{i\theta}) = \lambda\phi(e^{i\theta}) \qquad \text{for all} \quad \theta \in [0, 2\pi].$$

The reproducing kernel property implies

$$\begin{aligned}(\tau'\phi)(e^{i\theta}) &= ((\tau'\phi)(\cdot), K(\cdot, e^{i\theta}))_{H^2}\\ &= (\phi(\cdot), \tau(K(\cdot, e^{i\theta})))_{L_2(E,\mu)}\\ &= \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it})\,\overline{K(e^{it}, e^{i\theta})}\,|B(e^{it})|^2\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, e^{it})\,\phi(e^{it})\,|B(e^{it})|^2\,dt.\end{aligned}$$

Therefore (2.2) is equivalent to the following integral equation:

$$\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, e^{it})\,\phi(e^{it})\,|B(e^{it})|^2\,dt = \lambda\phi(e^{i\theta}). \tag{2.3}$$

Theorem 2 will be proved if we manage to determine the largest eigenvalue of the last equation. For this purpose we observe that in view of (2.1) the function

$$\psi(e^{i\theta}) = \frac{1}{2i}(e^{in\theta} - e^{-in\theta}) = \sin(n\theta)$$

is an eigenfunction for the problem

$$\frac{1}{2\pi}\int_0^{2\pi} K(e^{i\theta}, e^{it})\,\psi(e^{it})\,dt = \frac{1}{q^{2n}+q^{-2n}}\,\psi(e^{i\theta}).$$

Repeating the same analysis as that above for the operator $T'\colon L_2(E) \to H^2$ instead of τ', we see that

$$(T'\psi)(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z, e^{it})\,\psi(e^{it})\,dt \qquad \text{for all} \quad z \in \Omega.$$

The identity theorem for holomorphic functions implies

$$(T'\psi)(z) = \frac{1}{q^{2n}+q^{-2n}}\,\psi(z) \qquad \text{for all} \quad z \in \Omega.$$

Consequently

$$(\psi, Tf)_{L_2(E)} = \frac{1}{q^{2n}+q^{-2n}}\,(\psi, f)_{H^2} \qquad \text{for all} \quad f \in H^2.$$

Let us fix a point $w \in \Omega$ and choose in particular

$$f_0(z) = K(z, w)\, B(z)\, \overline{B(w)}.$$

Since $\overline{B(z)} = 1/B(z)$ for $z \in \partial\Omega$, we obtain

$$(\psi, f_0)_{H^2} = (\psi(\cdot)/B(\cdot), K(\cdot, w)\, \overline{B(w)})_{H^2}.$$

Since the zeros of ψ and B coincide, $g = \psi/B$ is a well defined function in H^2 and therefore

$$(\psi, f_0)_{H^2} = \psi(w).$$

On the other side,

$$\begin{aligned}(\psi, Tf_0)_{L_2(E)} &= \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{it})\, \overline{K(e^{it}, w)}\, \overline{B(e^{it})}\, B(w)\, dt \\ &= B(w) \frac{1}{2\pi} \int_0^{2\pi} K(w, e^{it})\, g(e^{it})\, |B(e^{it})|^2\, dt.\end{aligned}$$

Combining the last two results with the special choice $w = e^{i\theta}$ yields that

$$\frac{1}{2\pi} \int_0^{2\pi} K(e^{i\theta}, e^{it})\, g(e^{it})\, |B(e^{it})|^2\, dt = \frac{1}{q^{2n} + q^{-2n}}\, g(e^{i\theta}).$$

Hence g is an eigenfunction for Eq. (2.3). By definition g is zero free in Ω and real on E. In particular, g cannot change its sign on E.

Finally, let us take into account that the symmetric kernel $K(e^{i\theta}, e^{it})$ is positive on E as is proved in Pinkus [7, Chap. III. 4]. Now it is a familiar result that if an integral operator with a positive symmetric kernel possesses an eigenfunction without sign changes, then the corresponding eigenvalue must be the largest eigenvalue of the problem; for a proof we refer the reader to Melkman and Micchelli [6]. Consequently, $1/(q^{2n} + q^{-2n})$ must be the largest eigenvalue of the integral equation (2.3). This completes the Proof of Theorem 2.

Proof of Theorem 3. The starting point for the proof of Theorem 3 is the observation that in view of the Hilbert space structure of H^2 the operator $f \mapsto f - \sigma_{2n}(f)$ is the orthogonal projection of H^2 onto the subspace $V^{2n} = \{f \in H^2 : f(\zeta_j) = 0, j = 1, \ldots, 2n\}$. In order to get an explicit representation for $f - \sigma_{2n}(f)$, we introduce the kernel

$$K_{2n}(z, \zeta) := B(z)\, \overline{B(\zeta)}\, K(z, \zeta)$$

where $K(z, \zeta)$ is the reproducing kernel of H^2.

Set

$$Q_{2n}(f)(\zeta) := (f(\cdot), K_{2n}(\cdot, \zeta))_{H^2}.$$

Since $(f(\cdot), K_{2n}(\cdot, \zeta_j)) = (f(\cdot), B(\cdot)\, K(\cdot, \zeta_j))\, B(\zeta_j) = 0$, we have $Q_{2n}(f)(\zeta_j) = 0$ for any $f \in H^2$ and $j = 1, ..., 2n$. If $f \in V^{2n}$, then $f/B \in H^2$ and using that $\overline{B(z)} = 1/B(z)$ on $\partial\Omega$, we obtain that $(f(\cdot), K_{2n}(\cdot, \zeta)) = (f(\cdot)/B(\cdot), K(\cdot, \zeta))\, B(\zeta) = f(\zeta)$. Thus $Q_{2n}(f) = f$ for $f \in V^{2n}$. Finally, Q_{2n} is selfadjoint, since K_{2n} is Hermitian. Consequently, Q_{2n} is the orthogonal projection of H^2 onto V^{2n}, i.e., $Q_{2n}(f) = f - \sigma_{2n}(f)$.

In order to evaluate $(f(\cdot), K_{2n}(\cdot, \zeta))_{H^2}$ we first express $K(z, \zeta)$ in terms of elliptic functions:

$$\begin{aligned} K(z, \zeta) &= \sum_{s=-\infty}^{\infty} \frac{z^s \bar{\zeta}^s}{q^{2s} + q^{-2s}} = \frac{1}{2} + \sum_{s=1}^{\infty} \frac{\cos((s/i)(\log(z) + \log(\bar{\zeta})))}{\cosh(2s\beta)} \\ &= \frac{K}{\pi}\, dn\left(\frac{K}{\pi i}(\log(z) + \log(\bar{\zeta})), k\right) \end{aligned}$$

where K and K' are complete elliptic integrals of the first kind with moduli k and $k' = \sqrt{1-k^2}$ which satisfy the equation

$$\frac{\pi K'}{2K} = \beta = \log(1/q).$$

Here and in the following we use the standard notation for the Jacobian elliptic functions (see for example Bateman [1]). In particular, for $z \in \partial\Omega$, i.e. $|z| = q^{\mp 1} = e^{\pm\beta}$, we have

$$\begin{aligned} \overline{K(z, \zeta)} = K(\zeta, z) &= \frac{K}{\pi}\, dn\left(\frac{K}{\pi i}(\log(\zeta) - \log(z) \pm 2\beta), k\right) \\ &= \frac{K}{\pi}\, dn\left(\frac{K}{\pi i}(\log(\zeta) - \log(z)) \mp iK', k\right) \\ &= \pm \frac{K}{\pi i} \frac{cn((K/\pi i)(\log(z) - \log(\zeta)), k)}{sn((K/\pi i)(\log(z) - \log(\zeta)), k)}. \end{aligned}$$

It is easy to show that

$$(f, g)_{H^2} = \frac{1}{2\pi i} \int_{C_{1/q}} f(z)\, \overline{g(z)}\, \frac{dz}{z} - \frac{1}{2\pi i} \int_{C_q^-} f(z)\, \overline{g(z)}\, \frac{dz}{z}$$

where $C_\rho = \{z \in \mathbb{C} : |z| = \rho\}$ and the minus in C_q^- indicates that C_q is traversed in the clockwise direction, while $C_{1/q}$ is traversed counter-clockwise.

Set $C = C_{1/q} \cup C_q^-$. Then

$$Q_{2n}(f)(\zeta) = B(\zeta) \frac{1}{2\pi i} \int_C \frac{K}{\pi i} \frac{f(z)}{B(z)} \frac{cn((K/\pi i)(\log(z) - \log(\zeta)), k)}{sn((K/\pi i)(\log(z) - \log(\zeta)), k)} \frac{dz}{z}.$$

The last integral can be evaluated by the residue theorem

$$\begin{aligned} &Q_{2n}(f)(\zeta) \\ &= f(\zeta) - \frac{K}{\pi i} B(\zeta) \sum_{j=1}^{2n} \frac{1}{\zeta_j B'(\zeta_j)} \frac{cn((K/\pi i)(\log(\zeta) - \log(\zeta_j)), k)}{sn((K/\pi i)(\log(\zeta) - \log(\zeta_j)), k)} f(\zeta_j). \end{aligned}$$

Since $Q_{2n}(f) = f - \sigma_{2n}(f)$, the minimal interpolant is given by

$$\sigma_{2n}(f)(\zeta) = \frac{K}{\pi i} B(\zeta) \sum_{j=1}^{2n} \frac{1}{\zeta_j B'(\zeta_j)} \frac{cn((K/\pi i)(\log(\zeta) - \log(\zeta_j)), k)}{sn((K/\pi i)(\log(\zeta) - \log(\zeta_j)), k)} f(\zeta_j).$$

What remains to be done is to express $B(\zeta)$ and $B'(\zeta_j)$ in terms of elliptic functions. B can be written in the form

$$B(\zeta) = k^n \prod_{j=0}^{2n-1} sn\left(\frac{K}{\pi i} \log(\zeta) - j\frac{K}{n}, k\right).$$

Using the first fundamental transformation of elliptic functions of degree $2n$ one can show that

$$B(\zeta) = -\sqrt{\lambda}\, sn\left(\frac{2n\Lambda}{\pi i} \log(\zeta), \lambda\right).$$

Here Λ is the complete elliptic integral of the first kind with modulus λ determined by the equation

$$\frac{\Lambda'}{\Lambda} = 2n \frac{K'}{K}.$$

Consequently,

$$B'(\zeta_j) = \sqrt{\lambda}(-1)^j \frac{2n\Lambda}{\pi i \zeta_j}.$$

Inserting these results in the above representation for $\sigma_{2n}(f)$ yields:

$$\sigma_{2n}(f)(\zeta) = \frac{K}{2nA} sn\left(\frac{2nA}{\pi i} \log(\zeta), \lambda\right) \sum_{j=1}^{2n} (-1)^{j+1}$$
$$\times \frac{cn((K/\pi i) \log(\zeta) - (j-1)(K/n), k)}{sn((K/\pi i) \log(\zeta) - (j-1)(K/n), k)} f(\zeta_j).$$

This completes the Proof of Theorem 3.

References

1. Bateman Manuscript Project, "Higher Transcendental Functions," Volume II, McGraw-Hill, New York, 1953.
2. S. D. Fisher, "Function Theory on Planar Domains: A Second Course in Complex Analysis," Wiley-Interscience, New York, 1983.
3. S. D. Fisher and C. A. Micchelli, Optimal sampling of holomorphic functions, *Amer. J. Math.* **106** (1984), 593–609.
4. S. D. Fisher and C. A. Micchelli, Optimal sampling of holomorphic functions II, *Math. Ann.* **273** (1985), 131–147.
5. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart, and Winston, New York, 1966.
6. A. A. Melkman and C. A. Micchelli, Spline spaces are optimal for L^2 n-widths, *Illinois J. Math.* **22** (1978), 541–564.
7. A. Pinkus, "n-Widths in Approximation Theory," Springer-Verlag, Berlin, 1985.
8. D. Sarason, The H^p spaces of an annulus, *Mem. Amer. Math Soc.* **56** (1965).